Journal of Applied Mechanics and Technical Physics, Vol. 35, No. 6, 1994

LONG WAVES IN AN EDDYING BAROTROPIC LIQUID

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UDC 532.591+517.958

The integro-differential equations of the theory of long waves which describe eddying flow of a barotropic liquid with a free boundary in an Euler–Lagrange system of coordinates are considered. For flows with a velocity profile that is monotonic with depth, the necessary and sufficient conditions for the system of equations of motion to be hyperbolic are obtained; these are the necessary conditions for the Cauchy problem to be well-posed. The velocities of the characteristics are determined and the characteristic form of the system is calculated. An example of the initial data for which the Cauchy problem is ill-posed is constructed.

The steady solutions of the system describing flow with a critical layer were investigated in [1], and particular simple-wave solutions were considered in [2].

1. Model of Long Waves. The solution of the initial boundary-value problem

$$
\begin{gathered}
u_T + uu_X + vu_Y + \rho^{-1}p_X = 0, \\
\varepsilon^2(v_T + uv_X + vv_Y) + \rho^{-1}p_Y = -1, \ 0 \leq Y \leq h(X, T), \\
\rho_T + u\rho_X + v\rho_Y + \rho(u_X + v_Y) = 0, \ -\infty < X < \infty, \\
\rho = \rho(p), \ \rho'(p) > 0, \\
\rho(X, Y, 0) = \rho_0(X, Y), \ u(X, Y, 0) = u_0(X, Y), \\
v(X, Y, 0) = v_0(X, Y), \ h(X, 0) = h_0(X), \ v(X, 0, T) = 0, \\
h_T + u(X, h, T)h_X = v(X, h, T), \ p(X,h,T) = p_0 = \text{const}
\end{gathered}
\tag{1.1}
$$

describes the plane-parallel motion of a layer of barotropic liquid with a free boundary $Y = h(X, T)$ over a smooth bottom in a gravitational field of force. Here

$$
\begin{gathered}
u_1 = (gH_0)^{1/2}u, \ v_1 = (gH_0)^{1/2}H_0L_0^{-1}v, \ p_1 = R_0gH_0p, \\
\rho_1 = R_0\rho, \ X_1 = L_0X, \ Y_1 = H_0Y, \ T_1 = L_0(gH_0)^{-1/2}T
\end{gathered}
$$

are the dimensional components of the velocity vector, the pressure, density and Cartesian coordinates in a plane and in time, u, v, p, ρ, X, Y, T are the corresponding dimensionless quantities, H_0 and L_0 are the characteristic vertical and horizontal scales, g is the acceleration due to gravity, and R_0 has the dimensions of density. The specified functions u_0, v_0, ρ_0, h_0 are determined for $0 \leq Y \leq h_0(X)$, $X \in (-\infty, \infty)$.

The theory of long waves (the theory of shallow water) arises in the passage to the limit $\varepsilon = H_0L_0^{-1} \to 0$. Here, the second equation of system (1.1) becomes the hydrostatic law of the pressure distribution with depth:

$$
p_Y = -\rho(p) \quad (p(X, h, T) = p_0). \tag{1.2}
$$

Integrating (1.2) and the equation of continuity we obtain the relations

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 6, pp. 17-26, November-December, 1994. Original article submitted November 22, 1994.

$$p = f(h - Y) \quad \left(f\left(\int_{p_0}^{p} \rho^{-1}(\xi)d\xi\right) = p\right),$$

$$\rho = f'(h - Y),\ v = -\rho^{-1} \int_0^Y (\rho_T + (u\rho)_X)dY. \tag{1.3}$$

After some reduction we obtain the problem of finding u(X, Y, T) and h(X, T):

$$\begin{gathered} u_T + uu_X + vu_Y + h_X = 0,\ 0 \leqslant Y \leqslant h(X, T), \\ f'(h)h_T + \left(\int_0^h f'(h - Y)u(X, Y, T)dY\right)_X = 0, \\ u(X, Y, 0) = u_0(X, Y),\ h(X, 0) = h_0(X) \end{gathered} \tag{1.4}$$

(p, ρ, v are given in (1.3)).

For the special class of eddy-free motions (in the long-wave approximation the fact that there are no eddies is equivalent to the condition $u_Y = 0$), Eqs. (1.4) reduce to the analog of one-dimensional gas dynamics:

$$\begin{gathered} \eta(u_T + uu_X) + P_X = 0,\ \eta_T + (u\eta)_X = 0 \\ \left(\eta = f(h) - p_0,\ P = \int_0^h (f(\xi) - p_0)d\xi\right). \end{gathered} \tag{1.5}$$

Here the equation of state of the "gas" has the form

$$P = P(\eta) = \int_{p_0}^{\eta + p_0} (\xi - p_0)(\rho(\xi))^{-1}d\xi$$

($P'(\eta) > 0$ when $\eta > 0$). System (1.5), which describes the propagation of long waves in eddy-free flow is a hyperbolic system of differential equations.

In the general case of eddying flow, when $u_Y \neq 0$, the integro-differential equations (1.4) can be reduced to an infinite system of differential equations for the momenta:

$$\begin{gathered} A_{nT} + A_{n+1X} + nA_{n-1}\left(\int_{p_0}^{A_0 + p_0} \rho^{-1}(\xi)d\xi\right)_X = 0 \\ \left(A_n = \int_0^h \rho u^n dY,\ n = 0, 1, 2, \ldots\right). \end{gathered} \tag{1.6}$$

It is difficult to investigate the conditions under which the infinite system of equations of the form (1.6) is hyperbolic, since a general theory of such systems has not been developed. On the other hand, we can convert the problem with free boundary (1.4) to a Cauchy problem for integro-differential equations of special form in a fixed region, and we can use the method of analyzing the characteristic properties of integro-differential equations [3].

Using the replacement of variables [3]

$$X = x,\ T = t,\ Y = \Phi(x, t, \lambda)\ (0 \leqslant \lambda \leqslant 1), \tag{1.7}$$

where $\Phi(x, t, \lambda)$ is the solution of the Cauchy problem

$$\Phi_t + u(x, \Phi, t)\Phi_x = v(x, \Phi, t),\ \Phi(x, 0, \lambda) = \lambda h_0(x), \tag{1.8}$$

the region of flow is mapped into the strip $0 \leq \lambda \leq 1$, $-\infty \leq x \leq \infty$.

By virtue of (1.4) and (1.8) for the functions

$$u(x, t, \lambda),\ H(x, t, \lambda) = \rho(x, t, \lambda)\Phi_\lambda(x, t, \lambda)\ (H > 0),$$

defined in this strip, we obtain the Cauchy problem

$$\begin{gathered} u_t + uu_x + (\rho(p^0))^{-1}\int_0^\lambda H_x d\nu = 0,\ H_t + (uH)_x = 0, \\ u(x, 0, \lambda) = u_0(x, \lambda h_0(x)),\ H(x, 0, \lambda) = \rho_0(x, \lambda h_0(x))h_0(x). \end{gathered} \tag{1.9}$$

The pressure $p(x, t, \lambda)$ can then be found from the formula

$$p(x, t, \lambda) = p_0 + \int_\lambda^1 H(x, t, \nu)d\nu.$$

In (1.9) we have denoted by p^0 the pressure on the bottom ($p^0 = p(x, t, 0)$). If the functions u, H and p are defined, the equation of state $\rho = \rho(p)$ enables us to find the density, while the relations

$$\Phi_\lambda = \rho^{-1}H,\ \Phi(x, t, 0) = 0,\ \Phi_t + u\Phi_x = v$$

define the function Φ and the vertical component of the velocity. When $\Phi_\lambda \neq 0$, the replacement of variables (1.7) is reversible, which enables us to obtain the solution $u(X, Y, T)$, $h(X, T) = \Phi(X, T, 1)$ of problem (1.4). Hence, the problem of eddying flows with a free boundary (1.1) reduces to problem (1.9) in a fixed region.

2. Condition for the System of Equations (1.9) to be Hyperbolic. The question of whether the Cauchy problem is well-posed for system (1.9) for arbitrary initial data is still an open one. We also do not know whether Eqs. (1.9) possess such an important qualitative property, which characterizes wave processes, as finiteness of the velocity of propagation of perturbations. The problem arises of the correct formulation of the boundary conditions for modeling flows in bounded regions. For a system of first-order partial differential equations the solution of these problem involves investigating the properties of hyperbolicity of the system and the behavior of its characteristics. In this section we will investigate the conditions for Eqs. (1.9) to be hyperbolic. The basis of the analysis is a generalization of the idea of hyperbolicity and the determination of the characteristics assumed for systems of equations with operator coefficients in [3].

The system of equations (1.9) has the form

$$U_t + AU_x = 0,\ U = \begin{pmatrix} u(x, t, \lambda) \\ H(x, t, \lambda) \end{pmatrix}, \tag{2.1}$$

where A is an operator acting on the vector-function $f = (f_1, f_2)^T$ according to the rule

$$Af = \begin{pmatrix} uf_1 + \sigma\int_0^1 f_2 d\lambda \\ hf_1 + uf_2 \end{pmatrix},\ \sigma = [\rho(p_0 + \int_0^1 H d\nu)]^{-1}.$$

The characteristic curve of the system of equations (2.1) $x = x(t)$ is defined [3] by the differential equation $x'(t) = k(x, t)$ (k is the eigenvalue of the operator A^*). The equation

$$(F,(A - kI)\varphi) = 0 \tag{2.2}$$

for the linear vector-functional $F = (F_1, F_2)$, which acts on an arbitrary infinitely differentiable vector-function of the variable λ (which depends on x and t as on the parameters), then has a nontrivial solution in the class of locally integrable or generalized functions (the functions u and H are assumed to be infinitely differentiable and (F, φ) denotes the results of the action of the

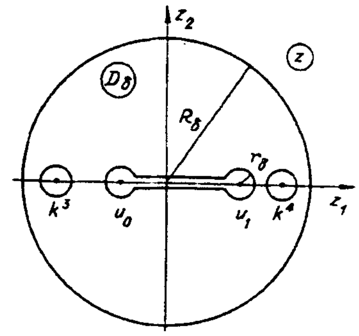

Fig. 1

functional F on the trial function $\varphi = (\varphi_1, \varphi_2)^T)$). By the action of F on Eqs. (2.1) we obtain the following relation on the characteristic:

$$(F, U_t + kU_x) = 0. \tag{2.3}$$

System (2.1) will be hyperbolic if all the eigenvalues k are real and the set of relations on the characteristics (2.3) is equivalent to Eqs. (2.1).

Since the trial functions φ_1, φ_2 are independent we obtain the following equations from (2.2):

$$(F_1,(u - k)\varphi_1) + (F_2, H\varphi_1) = 0, \; (F_2,(u - k)\varphi_2) + \sigma \int_0^1 \varphi_2 d\nu (F_1, 1) = 0.$$

By virtue of these equations, the action of F_2 on the trial function φ is defined if the action F_1 is specified:

$$(F_2, \varphi) = - (F_1,(u - k)H^{-1}\varphi). \tag{2.4}$$

The functional F_1 and the quantity k are found by solving the eigenvalue problem

$$(F_1,(u - k)^2 H^{-1}\psi) - \sigma \int_0^1 \psi d\nu (F_1, 1) = 0 \tag{2.5}$$

(ψ is an arbitrary trial function). To construct the solutions of Eq. (2.5) we will consider the set of numbers k in the complex plane outside the section of values of the function u: $k \neq u(x, t, \lambda)$ (x and t are fixed), $\lambda \in [0, 1]$. It follows from (2.5) that

$$(F_1, \psi) = \sigma \int_0^1 \frac{\psi H d\nu}{(u - k)^2} (F_1, 1). \tag{2.6}$$

Nontrivial solutions of Eq. (2.5) exist for $k = k^i$, which satisfy the characteristic equation

$$\chi(k^i) = 1 - \sigma \int_0^1 \frac{H d\nu}{(u - k^i)^2} = 0. \tag{2.7}$$

For $k = k^i$ the action of the functional F_1^i on the trial function ψ is specified by (2.6), where we must put $(F_1^i, 1) = 1$ (F_1^i is defined apart from a factor which is independent of x and t). Henceforth we will consider flows with a monotonic velocity profile: $u_\lambda \neq 0$. For smooth flow it is sufficient to require the condition $u_\lambda \neq 0$ at the initial instant of time.

Suppose $k = k^\lambda$ is one of the values of horizontal velocity: for a certain λ $k^1(x, t) = u(x, t, \lambda)$. For each k^λ we determined the action of the functionals $F_1^{\lambda 1}$, $F_1^{\lambda 2}$ on $\psi(\nu)$:

$$(F_1^{\lambda 1}, \psi) = - \psi'(\lambda), \; (F_1^{\lambda 2}, \psi) = \psi(\lambda) + \sigma \int_0^1 \frac{H(\nu)(\psi(\nu) - \psi(\lambda))d\nu}{(u(\nu) - u(\lambda))^2}. \tag{2.8}$$

Here the integral is evaluated in the sense of the principal value; we have omitted the dependence of the functions on the variables x and t in the notation for the multiplicity of the notation. A direct evaluation shows that (2.5) holds if $k = k^{\lambda}(x, t)$, and $F_1 = F_1^{\lambda 1}$ or $F_1 = F_1^{\lambda 2}$. The characteristic equation (2.7) has two roots ($k = k^3$, $k = k^4$) on the real axis outside the range of values of u(x, t, λ). To fix our ideal we will assume that $u_\lambda > 0$ (the case when $u_\lambda < 0$ is exactly analogous). Since $\chi(k) \to 1$ when $|k| \to \infty$, $\chi(k) \to -\infty$ when $k \to u_0 = u(x, t, 0)$ and $k \to u_1 = u(x, t, 1)$, the roots $k = k^3 < u_0$, $k = k^4 > u_1$ are obtained. The inequality $\chi'(k) \neq 0$ is satisfied when $k > u_1$ and $k < u_0$, and hence there are no other real roots outside the section $[u_0, u_1]$.

By (2.4) the functionals $F_2^{\lambda 1}$, $F_2^{\lambda 2}$, F_2^i (i = 3, 4) act on the trial function $\varphi(\nu)$ according to the rule

$$(F_2^{\lambda 1}, \varphi) = u_\lambda(\lambda) H^{-1}(\lambda)\varphi(\lambda), \ (F_2^i, \varphi) = -\sigma \int_0^1 \frac{\varphi(\nu)d\nu}{u(\nu) - k^i},$$
$$(F_2^{\lambda 2}, \varphi) = -\sigma \int_0^1 \frac{\varphi(\nu)d\nu}{u(\nu) - u(\lambda)}.$$

The conditions for the system of equations (1.9) to be hyperbolic will be formulated in terms of $\chi^{\pm}(x, t, \lambda) = \chi^{\pm}(u(\lambda))$ – the limiting values of the analytic function $\chi(z)$ in the upper and lower half-planes in the section [u(x, t, 0), u(x, t, 1)]:

$$\chi^{\pm}(u(\lambda)) = 1 + \sigma\omega_1^{-1}(u_1 - u(\lambda))^{-1} - \sigma\omega_0^{-1}(u_0 - u(\lambda))^{-1}$$
$$- \sigma \int_0^1 \left(\frac{1}{\omega}\right)_\nu \frac{d\nu}{u(\nu) - u(\lambda)} + \pi i \left(\frac{1}{\omega}\right)_\lambda \frac{\sigma}{u_\lambda}.$$

Here $\omega = u_\lambda H^{-1}$; the limits 0 and 1 correspond to the values of the functions when λ = 0 and 1; i is the square root of −1 (when calculating the limiting values we used (2.7) and the Sokhotskii–Plemel formula [4]).

The following lemma holds for a monotonic profile of the velocity.

Lemma 2.1. Suppose the functions u(x, t, λ), H(x, t, λ) satisfy the conditions

$$\chi^{+} \neq 0, \ k = \Delta\arg\chi^{+}(u)(\chi^{-}(u))^{-1} = 0. \tag{2.9}$$

Then Eq. (2.7) has only real roots in (2.9) arg f(u) is the increment of the argument of the complex-valued function f when λ changes from 0 to 1).

Proof. Consider the region D_δ in the plane of the complex variable z, bounded by a circle or radius R_δ with center at the origin of coordinates, circles of radii r_δ with centers at the points k^3, k^4, u_0, u_1 and sections of parallel straight lines a distance δ from the real axis ($r_\delta \to 0$, $R_\delta \to \infty$ when $\delta \to 0$, see the figure). According to the principle of the argument [4], the increment in the argument of the function $\chi(z)$ along the boundary of the region D_δ, normalized to 2π, is equal to the number of zeros of the function $\chi(z)$ in the region D_δ ($\chi(z)$ has no poles in the region D_δ). Since $\chi(z)$ has first-order zeros at the points $z = k^i$ (i = 3, 4) and first-order poles at the points $z = u_0$ and $z = u_1$, the increments of the argument in small neighborhoods will, in sum, be equal to zero and the condition for there to be no roots in the section D_δ as $[u_0, u_1]$.

Note 2.1. Any complex root $k = k_1 = ik_2$ ($k_2 \neq 0$) of Eq. (2.7) belongs to the subregion

$$(k_1 - r_0)^2 + k_2^2(1 + \sigma^{-1}(f(h) - p_0)^{-1}k_2^2) < r^2$$
$$(r = 2^{-1}(u_1 - u_0), \ r_0 = 2^{-1}(u_1 + u_0))$$

of the circle $|k - r_0| < r$ (the analog of Howard's theorem). In addition $u_0 < k_1 < u_1$.

These inequalities are a consequence of the relations which follow from (2.7) when split into imaginary and real parts,

$$\int_0^1 \frac{(u - k_1)Hd\nu}{|u - k|^4} = 0,$$

$$\int_0^1 \frac{((u-k_1)^2 - k_2^2 - \sigma^{-1}(f(h) - p_0)^{-1}|u-k|^4)H\,d\nu}{|u-k|^4}$$

$$= \int_0^1 \frac{((u-r_0)^2 - (k_1 - r_0)^2 - k_2^2 - \sigma^{-1}(f(h) - p_0)^{-1}|u-k|^4)H\,d\nu}{|u-k|^4} = 0$$

and the obvious inequalities $(u - r_0)^2 \le r^2$, $|u - k|^4 \ge k_2{}^4$. Hence, if, during the evolution of the flow, complex roots appear for the first time in the equation $\chi(z) = 0$, and branch off from those points of the section of the real axis $[u_0, u_1,]$, where the equality $\chi^{\pm} = 0$ is satisfied. the condition $\chi^+ \neq 0$ in (2.9) excludes the neutral case and thereby ensures that the roots of Eqs. (2.7) are real not only for specified profiles of u and H but also for fairly small smooth perturbations of these profiles.

The system of relations on the characteristics obtained by the action of the functions $F^{\lambda 1}$, $F^{\lambda 2}$, F^i on Eqs. (1.9) has the form

$$\begin{gathered} u_{\lambda t} + uu_{\lambda x} - u_\lambda H^{-1}(H_t + uH_x) = 0, \\ u_t + uu_x + \sigma \int_0^1 \frac{H(\nu)(u_t(\nu) + u(\lambda)u_x(\nu) - u_t(\lambda) - u(\lambda)u_x(\lambda))}{(u(\nu) - u(\lambda))^2} d\nu \\ - \sigma \int_0^1 \frac{H_t(\nu) + u(\lambda)H_x(\nu)}{u(\nu) - u(\lambda)} d\nu = 0, \\ \int_0^1 \frac{H(\nu)(u_t(\nu) + k^i u_x(\nu))d\nu}{(u(\nu) - k^i)^2} - \int_0^1 \frac{(H_t(\nu) + k^i H_x(\nu))d\nu}{(u(\nu) - k^i)} = 0 \ (i = 3, 4). \end{gathered} \tag{2.10}$$

Lemma 2.2. Suppose the functions S_1, $S_{1\lambda}$, S_2 satisfy the Hölder condition with respect to the variable λ, and the relations $(F^{\lambda 1}, S) = 0$, $(F^{\lambda 2}, S) = 0$, $(F^i, S) = 0$ $(t = 3, 4)$ are satisfied for the vector-functions $S = (S_1, S_2)^T$. Then $S = 0$.

Proof. Taking the first equation $S_{1\lambda} = u_\lambda H^{-1} S_2$ into account, the equation $(F^{\lambda 2}, S) = 0$ becomes

$$S_1(\lambda) - \sigma \int_0^1 \frac{1}{\omega(\nu)} \frac{\partial}{\partial \nu} \left(\frac{S_1(\nu) - S_1(\lambda)}{u(\nu) - u(\lambda)} \right) d\nu = 0. \tag{2.11}$$

It can be shown by direct substitution that $\varphi = \alpha_1(u - k^3)^{-1} + \alpha_2(u - k^4)^{-1}$ satisfies Eq. (2.11) (α_1, α_2 are parameters independent of λ). Hence, the function $S_0 = S_1 - \varphi$ is also a solution of Eq. (2.11) and, by an appropriate choice of α_1 and α_2, vanishes when $\lambda = 0$ and 1. Integration by parts in (2.11) gives the following equation for determining $S_0(\lambda)$:

$$S_0\{\mathrm{Re}\chi^+\} - \sigma \int_{u_0}^{u_1} \frac{\partial}{\partial \nu} \frac{1}{\omega(\nu)} (u_\nu^{-1}) \frac{S_0(\nu) du(\nu)}{u(\nu) - u(\lambda)} = 0. \tag{2.12}$$

By the general theory of singular integral equations [4]. Eq. (2.12) is uniquely solvable in the class of functions which satisfy the Hölder condition with respect to the variable ν (or $u(\nu)$) and vanishes at the ends of the interval $[u_0, u_1]$, if $\chi^+ \neq 0$ and

$$\varkappa = \Delta \arg \chi^+(u)(\chi^-(u))^{-1} = 0.$$

Then $S_0(\lambda) = 0$.

The relations $(F, S) = 0$ $(t = 3, 4)$ give two equations for determining α_1 and α_2:

$$\alpha_1 \Gamma_{33} + \alpha_2 \Gamma_{34} = 0, \ \alpha_1 \Gamma_{34} + \alpha_2 \Gamma_{44} = 0.$$

Here

$$\Gamma_{ij} = \int_0^1 \omega^{-1} \frac{\partial}{\partial \nu} ((u - k^i)^{-1}(u - k^j)^{-1}) d\nu.$$

It can be verified that $\Gamma_{ij} = 0$, $i \neq j$, if k^i, and k^j are roots of Eq. (2.7). Since $\Gamma_{ij} \neq 0$, we have $\alpha_1 = \alpha_2 = 0$, $S_1 = S_0 + \varphi = 0$, $S_2 = (u_\lambda)^{-1} HS_{1\lambda} = 0$. This proves the lemma.

The following theorem holds for flows with a monotonic velocity profile satisfying the condition $\chi^+ \neq 0$.

Theorem 2.1. Conditions (2.9) are necessary and sufficient for Eqs. (1.9) to be hyperbolic.

In fact, by Lemma 2.1, conditions (2.9) are sufficient for the roots of the characteristic equation to be real. The conditions on the characteristics (2.10) have the form of the relations of Lemma 2.2, where the left-hand sides of Eqs. (1.9) are taken as S_i. When the conditions of Lemma (2.2) are satisfied the equality $S_i = 0$ follows from (2.10), which proves that systems (1.9) and (2.10) are equivalent. The necessity of the second condition (2.9) in the class of functions which satisfy the condition $\chi^+ \neq 0$, follows from the principle of the argument [4] of the theory of analytic functions.

Note 2.1. The conditions for the functions u and H to be continuous can be made less strict, in which case the assertion proved remains true. The conclusion of Theorem 2.1 holds if u and ω are differentiable functions u_λ and ω_λ satisfy the Hölder condition with respect to the variable λ.

The hyperbolicity of the system of equations on the initial data $u(x, 0, \lambda)$, $H(x, 0, \lambda)$ is necessary for the Cauchy problem to be locally well-posed. For Eqs. (1.9), linearized on the stationary solution $u = u^0(\lambda)$, $H = H^0(\lambda)$, one can construct an example of the ill-posed nature of the formulation of the Cauchy problem if, for the u^0 and H^0 considered, Eq. (2.7) has complex roots. This solution describes the shear flow of a layer of depth h_0, in which case $u = u^0(yh_0^{-1})$, $v = 0$, $\rho = f'(h_0 - y)$, $y = \lambda h_0$, $H^0(\lambda) = f'(h_0(1 - \lambda))h_0$. The system for perturbations

$$\begin{aligned} u_t + u^0 u_x + \sigma^0 \int_0^1 H_x d\nu = 0, \\ H_t + u^0 H_x + H^0 u_x = 0 \end{aligned} \tag{2.13}$$

has a solution in the form of an exponential function, increasing with time

$$U = (u, H)^T = e^{-l}(\varphi_1(\lambda),\ \varphi_2(\lambda))^T \exp(il(x - kt)) \tag{2.14}$$

(l is an arbitrary real number and k is a complex root (Im k > 0) of Eq. (2.7) with $u = u^0$, $H = H^0$). As $l \to \infty$ $U(x, 0, \lambda) \to 0$, but $U(x, t, \lambda)$ does not approach zero, which indicates that the Cauchy problem is ill-posed when conditions (2.9) break down.

Note that in the hyperbolic case Eqs. (2.13) can be reduced to characteristic form, similar to (2.10), which enables the linearized system to be integrated in explicit form.

3. Analysis of the Conditions for Hyperbolicity. As shown above, the analysis of the hyperbolicity of the equations of long waves has a direct relationship to the investigation of the stability of steady flows with a velocity profile "frozen" with respect to x and t (the velocity profile of steady flow is identical with the velocity profile of unsteady flow for chosen fixed values of x and t). For a number of well-known criteria of stability of the stationary shear flows of an incompressible fluid in a layer with a specified upper boundary, we obtain the corresponding analogs from (2.9) (in this case the liquid, in general, is compressible and the upper boundary is free).

Taking into account the equalities

$$\omega = -(du/dp),\ \Delta\arg\chi^+(\chi^-)^{-1} = 2\Delta\arg\chi^+$$

conditions (2.9) can be represented in the form

$$Z = A + iB \neq 0,\ \varkappa = \Delta\arg(A + iB) = 0,$$

$$A = m\left(1 - \sigma\left(\frac{du}{dp}\right)_1^{-1}(u_1 - u)^{-1}\right.$$

$$\left. + \sigma\left(\frac{du}{dp}\right)_0^{-1}(u_0 - u)^{-1} + \sigma\int_{u_0}^{u_1} \frac{d}{du}\left(\left(\frac{du}{dp}\right)^{-1}\right)\frac{dU}{U - u}\right),$$

$$B = m\sigma\pi i \frac{d}{du}\left(\left(\frac{du}{dp}\right)^{-1}\right), \ m = (u_1 - u)(u - u_0) \tag{3.1}$$

(the differentiation and integration are carried out for fixed x and t). The contour Γ, described by the point Z when u varies from u_0 to u_1, begins and ends on the real semiaxis (when $u_\lambda > 0$ $Z(u_1) > 0$, $Z(u_0) > 0$, and when $u_\lambda < 0$ $Z(u_1) < 0$, $Z(u_0) < 0$).

The increment of the argument is determined by the number of rotations of the point Z around zero, multiplied by 2π, for the motion of Z along Γ. If $d^2u/dp^2 \neq 0$, then Γ lies entirely in the upper or lower half-plane and Z undergoes no rotations around zero. Then $\varkappa = 0$ and the hyperbolicity conditions are satisfied (the analog of the classical criterion of Rayleigh stability). If $du/dp < 0$, and d^2u/dp^2 vanishes at the unique point $u = u_* \in (u_0, u_1)$, the condition $\varkappa = 0$ is satisfied if $A(u_*) > 0$ (the analog of the Rozenbluth and Simon criterion [5]). When this condition is satisfied the zero does not fall in the interior of Γ. Note that if the sign of d^2u/dp^2 changes from negative to positive at the point u_*, the inequality $A(u_*) > 0$ is automatically satisfied. In fact, the first terms in (3.1) for A are positive while the integral term can be represented in the form of two positive terms:

$$-\int_{u_0}^{u_*}\left(\frac{du}{dp}\right)^{-3}\frac{d^2u}{dp^2}\frac{dU}{U - u_*} - \int_{u_*}^{u_1}\left(\frac{du}{dp}\right)^{-3}\frac{d^2u}{dp^2}\frac{dU}{U - u_*}$$

(the integrals converge since $d^2u/dp^2 = 0$ when $u = u_*$). The condition $A \neq 0$ also ensures that (1.9) is hyperbolic.

We will give an example of a one-parameter family of steady solutions in which a change in the type of system of equations (1.9) occurs when the parameter changes.

Suppose the equation of state of a barotropic medium has the form $\rho = \rho(p) = bp^{1/2}$ (b = const), flow occurs in a layer of depth $h = 9(2b)^{-1}p_0^{2/3}$, and the pressure p in the layer and the horizontal velocity are defined by the formulas

$$p = (4p_0^{2/3} - 2 \cdot 3^{-1}by)^{3/2}, \ u = 4p_0^{1/3}a\upsilon(y) \ (a > 0),$$

where the function v(y) is the root of the equation

$$y = 6b^{-1}p_0^{2/3}(1 - (1 - 2(6^{-1}(2\upsilon - 1)^3 + 6^{-1} + 5(48)^{-1}\upsilon))^{2/3}),$$

which vanishes when $y = 0$. The following equation is satisfied for this solution:

$$dp/d\upsilon = -16p_0((2\upsilon - 1)^2 - 5(48)^{-1}) \tag{3.2}$$

($d^2p/dv^2 = 0$ when $v = 2^{-1}$). The integral in (3.1) can be evaluated in the form

$$A = 16p_0^{2/3}a^2(1 - \upsilon)\upsilon[1 + (32a^2b)^{-1}(53(3(1 - \upsilon))^{-1} + 53(3\upsilon)^{-1} - 128 - 64(2\upsilon - 1)\ln(\upsilon^{-1} - 1))].$$

When $v = 2^{-1}$, $A > 0$ when $a > a_k$ and $A < 0$ when $a < a_k$, where the critical value $a_k = (331/96)^{1/2} \cdot b^{-1}$. Hence, when $a < a_k$ Eq. (2.7) has complex roots. This example shows that instability of the steady flow is possible for flow velocities as low as desired. (The criterion of stability is formulated in terms of the function dp/du, and it approaches infinity as $a \to 0$.) If we assume that a is a function of x in the formulas derived above, we have an example of initial data for the system of equations (1.9) belonging to the region of hyperbolicity when $a(x) < a_k$, or an example of an ill-posed formulation of the Cauchy problem if the inequality $a(x) < a_k$ breaks down for certain x.

Note that when $\rho(p)$ = const the above analysis gives the conditions for the system of equations of long waves in an incompressible fluid to be hyperbolic. In the region of hyperbolicity the long-wave processes are characterized by a finite velocity of propagation of perturbations, where the velocity of propagation of the perturbation, by (2.7), depends nonlocally on the flow parameters, and is determined by the integrals of the flow parameters over the depth. Conditions (2.9) describe a region in which the long-wave model is applicable in the case of a monotonic velocity profile, and outside this region it is necessary to employ more complete models. On the basis of the above analysis it is natural to introduce the ideas of supercritical and subcritical steady flows. In supercritical flow, by (2.7)

$$1 - \sigma \int_{p_0}^{p} \frac{dp}{u^2} > 0,$$

while in subcritical flow the inverse inequality is satisfied. The behavior of the characteristics on the boundary affect the formulation of the boundary conditions, as in the case of the differential equations. In particular, on the boundary of the inflow of liquid one must formulate a different number of boundary conditions for subcritical and supercritical flows. It is also important to monitor that the conditions of hyperbolicity are satisfied in a numerical calculation of the flows. If the conditions of hyperbolicity are satisfied for $t = 0$ while the continuous solution of system (1.9) satisfies the condition $Z \neq 0$ in the range $0 \leq t \leq t_0$, the data for $t = t_0$ will also correspond to a region of hyperbolicity of Eqs. (1.9) (homotopic vector fields have the same rotation [6]).

This research was carried out with financial support from the Russian Fund for Fundamental Research (Project 93-013-17621).

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